How to Quickly Foil a Thief's Plan: An Elementary Algorithm for the Counterfeit Coin Problem

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Imagine yourself a tradesman. A thief walks into your store and says "Here are 39 gold coins. Give me my money's worth!" From his shifty look, you suspect he may be cheating you. Indeed he is: one of the 38 coins is not golden! But how do you tell which one? You go back into a separate room and begin inspecting the coins. If you stay there too long, the thief will figure out you are up to something and resort to ...extreme measures!

This problem is known as the counterfeit coin problem, and we are about to discover the algorithm for finding the counterfeit coin, and even whether it weighs less or more than the others, in only four weighings on a balance scale. In general, we will show that if there are $(3^N - 3)/2$ or fewer coins, and we know that one of them is counterfeit (i.e. it weighs either slightly less or slightly more than the others), then we can figure out which coin is different and whether it weighs less or more than the others by using only N weighings on a standard balance scale that only tells which side, if any, is heavier.

Throughout this note we will call the standard coin S and use the notation $A \leq B$, for example, when coin A weighs at most as much as coin B. Finally, if we find, for example, that A is the counterfeit coin and that it weighs less than the standard coin, we write the solution to the counterfeit problem as $(A, <)$.

Lemma 1. *Given 3^N coins, one of which is counterfeit, and a nonstrict*

comparison of each of them to a standard coin S (i.e. for each such coin A, we know either $A \geq S$ or $A \leq S$), we can solve the counterfeit coin problem using N weighings.

Proof. The proof is by induction on N. Suppose $N = 1$, i.e. we have three coins A, B, C. Without loss of generality we have four cases: either all three coins weigh at least as much as S, only two weigh at least as much as S, only one, or none. The last two cases are similar to the first two: just flip the inequality signs. If $A \geq S, B \geq S$, and $C \geq S$ then weigh A against B. If $A > B$ then the solution is $(A, >)$; if $A < B$ then it is $(B, >)$; if $A = B$ then $(C, >)$. Now, if $A \geq S, B \geq S$, but $C \leq S$ then, again, weigh A against B. The only difference is, if $A = B$ then the solution is $(C, <)$.

Now suppose it is true up to $N - 1$ for some N. We now have x coins $A_1, A_2, \ldots, A_x$ which weigh at least as much as S (i.e. $\forall k \leq x A_k \geq S$) and $3^N - x$ coins $B_1, B_2, \ldots, B_{3^N - x}$ which weigh at most as much as S. We can assume without loss of generality that $x > 3^N/2$ since otherwise we just switch all inequality signs. Now if $x \geq 2 \cdot 3^{N-1}$ then we weigh 3^{N-1} A coins against 3^{N-1} other A coins. If the scales balance, we know the rest of the 3^{N-1} coins have the counterfeit coin and we find the solution by induction. If the scales tip, we use induction on the side which weighs more. Finally, if $x < 2 \cdot 3^{N-1}$ then we weigh $(3^{N-1} + 1)/2$ A coins and $(3^{N-1} - 1)/2$ B coins against the same amounts of each. If the scales balance, we again simply use induction on the remaining 3^{N-1} coins. If they tip, we use induction on the A coins from the heavier side and the B coins from the lighter side. $\square$

The reader will note that if there are fewer than 3^N coins in the above lemma, we can still solve the problem in N weighings by induction: if we have $M > 2$ coins instead, we weigh $\lfloor M/3 \rfloor$ coins against $\lfloor M/3 \rfloor$ others, making sure we have at least as many A coins as B coins on each side of the scale. This remark holds for the theorem to follow, as well.

Theorem 1. *Assume $N \geq 2$. Given $(3^N - 3)/2$ coins, it is possible to solve the counterfeit coin problem in N weighings. Moreover, given one standard coin S in addition to $(3^N - 1)/2$ questionable ones, it is possible to solve the counterfeit coin problem for these $(3^N - 1)/2$ coins in N weighings.*

Proof. Again, the proof is by induction. The case $N = 1$ is trivial, but the case $N = 2$ is a fun exercise. Now assume the theorem is true up to $N - 1$.

To solve the problem for $(3^N - 3)/2$ coins, we first divide this number into three equal parts and weigh one part against another. Note that

$$\frac{1}{3}\frac{3^N - 3}{2} = \frac{3^{N-1} - 1}{2}$$

so if the scales balance, we use induction (the second part of the theorem) on the remaining third of the coins. If they tip, then we have a nonstrict comparison between $3^{N-1} - 1$ coins and the standard: the coins on the heavier side weigh at least as much as the standard, and those on the other side weigh at most as much. By the lemma, this subproblem is solvable in $N - 1$ weighings.

The solution for $(3^N - 1)/2$ coins given one more standard coin S is quite similar: we first perform the same weighing as above but add one coin to the left side and S to the other. Thus we weigh $(3^{N-1} + 1)/2$ of the original coins against $(3^{N-1} - 1)/2$ of the original coins plus S. If the scales balance, we again use induction (the second part of the theorem) on the remaining coins. If they tip, we again have a nonstrict comparison between 3^{N-1} coins: those on the heavier side weigh at least as much as S, and those on the other - at most as much, so we use the lemma above. $\qquad\square$

Unfortunately, we cannot solve the counterfeit coin problem in N weighings if we have more than $(3^N - 1)/2$ coins (see wikipedia, for instance). By carefully reading the lemma and theorem above (and perhaps practicing a few times), the tradesman knows exactly how to find which coin is counterfeit, gives it back to the thief, and pays him for the other 38. He politely says "Thank you for your business," and the thief, dismayed by his politeness, simply leaves muttering.